Zeros of Lacunary Type of Polynomials

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Abstract In this paper we use matrix methods and Gereshgorian disk Theorem to present some interesting generalizations of some well-known results concerning the distribution of the zeros of polynomial. Our results include as a special case some results due to A .Aziz and a result of Simon Reich-Lossar.

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1. INTRODUCTION AND STATEMENT OF RESULTS

The following result due to Cauchy [4] is well known in the theory of the distribution of the zeros of a polynomial.

Theorem A. Let

\[ P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0 \]

be a polynomial of degree \( n \) then all the zeros of \( P(z) \) lie in the disk

\[ |z| < 1 + A. \]  \hspace{1cm} (1)

where \( A = \max |a_j|, j = 0, 1, 2, \ldots, n-1 \).

About forty years ago, in connection with Cauchy’s Classical result (Theorem A) Simon Reich proposed and among others Lossers [6] verified that if \( a_{n-1} = 0, Q > 1 \), then all the zeros of

\[ P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0, \]
Aziz [2] generalized the problem to lacunary polynomials and showed that the assertion (2), remains valid even if we do not assume that $Q > 1$. In fact he proved:

**Theorem B.** Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0,$$

$a_r \neq 0, 0 < r \leq n - 1$ be a polynomial of degree $n \geq 2$, with real or complex coefficients if

$$Q = \left\{ \max_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| \right\}^{\frac{1}{r}}$$

then all the zeros of $P(z)$ lie in the disk

$$|z| \leq Q + Q^2 + \ldots + Q^{r+1}$$

Where $0 \leq r \leq n - 1$. Other results of similar type were obtained among others by Alzer [1], Bell [3], Guggenheimer [5], Mohammad [7], Rahman [8], Walsh [10] (see also [9]).

As a generalization of Theorem B, we prove:

**Theorem 1.** Let

$$P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0$$

$a_r \neq 0, 0 \leq r \leq n - 1$ be a polynomial of degree $n \geq 2$, with real or complex coefficients if $t$ is any given positive number and

$$Q_t = \left\{ \max_{0 \leq j \leq r} \left| \frac{a_j}{a_n} t^{n-1} \right| \right\}^{\frac{1}{n}}$$

then all the zeros of $P(z)$ lie in the disk
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where 0 \leq r \leq n-1.

Taking \( t = 1 \), in equation (5), this reduces to Theorem B.

We next present the following result which provides an interesting refinement of Theorem 1.

**Theorem 2.** Let

\[ P(z) = a_r z^n + a_{r-1} z^{r-1} + \ldots + a_0 \]

\( a_r \neq 0, 0 \leq r \leq n - 1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if \( t \) is any given positive number and

\[ Q_i = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| t^{n-1} \right\}^{\frac{1}{n}} \]

then all the zeros of \( P(z) \) lie in the disk

\[ |z| \leq \frac{1}{t} \left\{ Q_i + \text{Max}(Q_i^2, Q_i^{r+1}) \right\} \]

(6)

where 1 \leq r \leq n-1. The following result immediately follows from Theorem 2 by taking \( t = 1 \):

**Corollary 1.** Let

\[ P(z) = a_r z^n + a_{r-1} z^{r-1} + \ldots + a_0 \]

\( a_r \neq 0, 0 \leq r \leq n - 1 \) be a polynomial of degree \( n \geq 2 \), with real or complex coefficients if \( t \) is any given positive number and

\[ Q_i = \left\{ \text{Max}_{0 \leq j \leq r} \left| \frac{a_j}{a_n} \right| t^{n-1} \right\}^{\frac{1}{n}} \]

then all the zeros of \( P(z) \) lie in the disk
where \( 1 \leq r \leq n - 1 \),

PROOF OF THE THEOREMS

Proof of Theorem 1. The companion matrix of the polynomial

\[ P(z) = a_n z^n + a_{r-1} z^{r-1} + \ldots + a_1 z + a_0 \]

\( a_r \neq 0 \) \( 0 \leq r \leq n - 1 \) of degree \( n \) is

\[
C = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & -a_0 t^{n-1} \\
\frac{Q_r}{t} & 0 & \ldots & 0 & \ldots & 0 & -a_r t^{n-2} \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & \frac{Q_r}{t} & \ldots & 0 & -a_r t^{n-r-1} \\
& & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & \ldots & \frac{Q_r}{t} & 0
\end{pmatrix}
\]

By hypothesis,

\[ Q_r = \max_{0 \leq j \leq n} \left| \frac{a_j}{a_n} \right| t^{n-j} \]

therefore,

\[ \left| \frac{a_j}{a_n} \right| t^{n-j} \leq Q_r^n \quad \text{for} \quad j = 0, 1, \ldots, r \quad \text{and} \quad Q_r \neq 0. \quad (7) \]

We take the matrix

\[
P = \text{diag} \left( \left\{ \frac{Q_r}{t} \right\}^{n-1}, \left\{ \frac{Q_r}{t} \right\}^{n-2}, \ldots, \left\{ \frac{Q_r}{t} \right\}, 1 \right)\]
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and form the matrix

\[
P^{-1}CP = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & \frac{-a_0 t^{n-1}}{a_n Q_{n-1}} \\
\frac{Q_t}{t} & 0 & \ldots & 0 & \ldots & 0 & \frac{-a_1 t^{n-2}}{a_n Q_{n-1}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \frac{Q_t}{t} & \ldots & 0 & \frac{-a_r t^{n-r-1}}{a_n Q_{n-r-1}} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & \frac{Q_t}{t} & 0
\end{pmatrix}
\]

Applying Gereshgorian Theorem to the columns of \( P^{-1} CP \) and noting (7), it follows that all the eigen values of the matrix \( P^{-1} CP \) lie in the circle

\[
|z| \leq \max \left\{ \frac{Q_t}{t}, \sum_{j=0}^{r} \frac{|a_j| t^{n-j-1}}{a_n |Q_{n-j-1}|} \right\}
\]

\[
\leq \frac{1}{t} \max \left\{ Q_t, \sum_{j=0}^{r} Q_j^{j+1} \right\}
\]

\[
= \frac{1}{t} \left\{ Q_t + Q_t^2 + \ldots + Q_t^{r+1} \right\}
\]

Since the matrix \( P^{-1} CP \) is similar to the matrix \( C \) and the eigen values of \( C \) are the zeros of the polynomial \( P(z) \), it follows that all the zeros of \( P(z) \) lie in the circle

\[
|z| \leq \frac{1}{t} \left\{ Q_t + Q_t^2 + \ldots + Q_t^{r+1} \right\}
\]

Which completes the proof of Theorem 1.

**Proof of Theorem 2.** The companion matrix of the polynomial

\[
P(z) = a_n z^n + a_r z^r + \ldots + a_1 z + a_0
\]

\( a_r \neq 0 \) \( 0 \leq r \leq n - 1 \) of degree \( n \) is given by
Proceeding similarly as in the proof of Theorem 1 and noting that

\[
P = \text{diag}\left\{ \left( \frac{Q_t}{t} \right)^{n-1}, \left( \frac{Q_t}{t} \right)^{n-2}, \ldots, \left( \frac{Q_t}{t} \right), 1 \right\}
\]

\[
Q_t = \left\{ \text{Max}_{0 \leq j \leq s} \left| \frac{a_j}{a_n} \right|^n t^{-j} \right\}
\]

It follows that the matrix

\[
P^{-1}CP = \begin{pmatrix}
0 & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & -\frac{a_0 t^{n-1}}{a_n Q_t^{n-1}} \\
\frac{Q_t}{t} & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & \ldots & 0 & -\frac{a_1 t^{n-2}}{a_n Q_t^{n-1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \ldots & \frac{Q_t}{t} & \ldots & 0 & \ldots & 0 & \ldots & 0 & -\frac{a_r t^{n-r-1}}{a_n Q_t^{n-r-1}} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\
0 & 0 & \ldots & 0 & \ldots & \frac{Q_t}{t} & 0 & 0 & 0 & 0 & \vdots
\end{pmatrix}
\]
Applying Gereshgorian Theorem to the columns of $P^{-1} CP$ and noting (7), it follows that all the eigen values of the matrix $P^{-1} CP$ therefore that of $C$ lie in the circle

$$|z| \leq \max_{1 \leq j \leq r} \left\{ a_n \left| \frac{t^{n-1}}{Q^r} \right|, a_j \left| \frac{t^{n-j-1}}{Q^r} \right| \right\}$$

$$\leq \frac{1}{t} \max_{1 \leq j \leq r} \left\{ Q, Q + Q^{r+1} \right\}$$

$$= \frac{1}{t} \left\{ Q + \max(Q^2, Q^{r+1}) \right\}$$

Since the matrix $P^{-1} CP$ is similar to the matrix $C$ and the eigen values of $C$ are the zeros of the polynomial $P(z)$, therefore we conclude that all the zeros of $P(z)$ lie in the circle denoted by (4). This proves Theorem 2 completely.

REFERENCES