Common Fixed Point Theorem For Mappings Satisfying (CLRg) Property

SAVITRI* AND NAWNEET HOODA

DCR University, Murthal (India)

*Email: savitrimalik1234@gmail.com

Received: February 19, 2015 | Revised: September 16, 2015 | Accepted: September 17, 2015

Abstract: The aim of this paper is to establish a common fixed point theorem for two pairs of mappings satisfying (CLRg) property.

Keywords: Common fixed point, complex-valued metric space, (CLRg) property, weakly compatible mappings.

1. INTRODUCTION

Fixed point theory has fascinated hundreds of researchers since 1922 with the celebrated Banach's fixed point theorem. This is a very active field of research at present. In 2011, Azam et al [6] introduced the concept of complex-valued metric space. Recently, Sintunavarat and Kumam [15] introduced the concept of (CLRg) property. Many results are proved on existence of fixed points in complex-valued metric spaces, see [1, 3-6, 8, 9, 11, 12, 14, 16, 17]. An interesting and detailed discussion on (CLRg) property is given by Babu and Subhashini [7].

In this paper, we use the concept of (CLRg) property and prove a common fixed point theorem for mappings satisfying (CLRg) property in complex-valued metric space.

2. PRELIMINARIES

Let \( \mathbb{C} \) be the set of complex numbers. Define a partial order \( \preceq \) on \( \mathbb{C} \) as follows:

\[
\begin{align*}
  z_1 \preceq z_2 & \quad \text{if} \quad \text{Re}(z_1) \leq \text{Re}(z_2), \quad \text{Im}(z_1) \leq \text{Im}(z_2), \\
  z_1 \not\preceq z_2 & \quad \text{if} \quad z_1 \neq z_2 \quad \text{and} \quad \text{either} \quad \text{Re}(z_1) < \text{Re}(z_2), \quad \text{Im}(z_1) < \text{Im}(z_2)
\end{align*}
\]
or \( \text{Re}(z_1) < \text{Re}(z_2) \), \( \text{Im}(z_1) = \text{Im}(z_2) \)

or \( \text{Re}(z_1) = \text{Re}(z_2) \), \( \text{Im}(z_1) < \text{Im}(z_2) \)

**Definition 2.1 ([6]).** Let \( X \) be a nonempty set such that the map \( d : X \times X \to \mathbb{C} \) satisfies the following conditions:

(c1) \( 0 \leq d(x, y) \) for all \( x, y \in X \) and \( d(x, y) = 0 \) iff \( x = y \);

(c2) \( d(x, y) = d(y, x) \) for all \( x, y \in X \);

(c3) \( d(x, y) \leq d(x, z) + d(z, y) \) for all \( x, y, z \in X \).

Then \( d \) is called a complex-valued metric on \( X \) and \( (X, d) \) is called complex-valued metric space.

**Definition 2.2 ([6]).** Let \( (X, d) \) be a complex-valued metric space and \( x \in X \). Then the sequence \( \{x_n\} \) is said to converge to \( x \) if for every \( 0 < c \in \mathbb{C} \), there is a natural number \( N \) such that \( d(x_n, x) < c \) for all \( n \in \mathbb{N} \).

We write it as \( \lim_{n \to \infty} x_n = x \).

**Definition 2.3 ([13]).** An element \( (x, y) \in X \times X \) is called coupled coincidence point of the mappings \( S : X \times X \to X \) and \( T : X \to X \) if

\[
S(x, y) = T(x), S(y, x) = T(y) .
\]

**Definition 2.4 ([10]).** An element \( x \in X \) is called common fixed point of the mappings \( S : X \times X \to X \) and \( T : X \to X \) if

\[
x = S(x, x) = T(x) .
\]

**Definition 2.5 ([2]).** The mappings \( S : X \times X \to X \) and \( T : X \to X \) are called \( w \)-compatible if \( TS(x, y) = S(Tx, Ty) \), whenever \( S(x, y) = Tx, S(y, x) = Ty \).

**Definition 2.6 ([10]).** The mappings \( S : X \times X \to X \) and \( T : X \to X \) are called commutative if \( TS(x, y) = S(Tx, Ty) \), for all \( x, y \in X \).

We note that the maps \( S : X \times X \to X \) and \( T : X \to X \) are weakly compatible if \( S(x, y) = T(x), S(y, x) = T(y) \) implies \( TS(x, y) = S(Tx, Ty), TS(y, x) = S(Ty, Tx) \) for all \( x, y \in X \).

**Definition 2.7 ([15]).** Let \( (X,d) \) be a metric space. Two mappings \( f : X \to X \) and \( g : X \to X \) are said to satisfy (CLRg) property if there exists a sequence \( \{x_n\} \subset X \) such that

\[
\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = g(p) \quad \text{for some} \quad p \in X .
\]
Definition 2.8 ([7]). Let \((X,d)\) be a metric space. Two mappings \(f : X \times X \to X\) and \(g : X \to X\) are said to satisfy (CLRg) property if there exist sequences \(\{x_n\}, \{y_n\} \subset X\) such that

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} g(x_n) = g(p),
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} g(y_n) = g(q), \quad \text{for some } p, q \in X.
\]

Definition 2.9 ([14]). The “max” function for the partial order relation “\(\preceq\)” defined by the

1. \(\max\{z_1, z_2\} = z_2\) if and only if \(z_1 \preceq z_2\),
2. If \(z_1 \preceq \max\{z_2, z_3\}\), then \(z_1 \preceq z_2\) and \(z_1 \preceq z_3\),
3. \(\max\{z_1, z_2\} = z_2\) if the only if \(z_1 \preceq z_2\) or \(|z_1| \preceq |z_2|\).

Example 2.1. Let \(X = [0, \infty)\) be a metric space under usual metric. Define mappings \(f : X \times X \to X\) and \(g : X \to X\) by

\[f(x, y) = x + y + 2, g(x) = 5 + x, \forall x, y \in X.\]

Let \(\{x_n\}\) and \(\{y_n\}\) be sequences in \(X\) where \(x_n = 3 + \frac{1}{n}\) and \(y_n = 3 - \frac{1}{n}\).

Since

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} (x_n + y_n + 2) = 8 = g(3),
\]

\[
\lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} \left(3 + \frac{1}{n}\right) = 3 + \frac{1}{n} + 5 = 8 = g(3)
\]

and

\[
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} (y_n + x_n + 2) = 8 = g(3),
\]

\[
\lim_{n \to \infty} g(y_n) = \lim_{n \to \infty} \left(3 - \frac{1}{n}\right) = 8 = g(3)
\]

So, the maps \(f\) and \(g\) satisfy (CLRg) property.

3. MAIN RESULTS

Theorem 3.1. Let \((X,d)\) be a complex valued metric-space and let \(f, g : X \times X \to X\) and \(\phi, \psi : X \to X\) are mappings such that
(1) \( d(f(x,y), g(u,v)) \leq p \max \{d(\phi x, \psi u), d(f(x,y), \phi x), d(g(u,v), \psi u), \\
\quad d(f(x,y), \psi u), d(g(u,v), \phi x) \} \)

for all \( x, y, u, v \in X \) and \( 0 < p < 1 \), (2) the pair \((f, \phi)\) and \((g, \psi)\) are weakly compatible. If the pair \((f, \phi)\) and \((g, \psi)\) satisfy (CLRg) property then \( f, g, \phi \) and \( \psi \) have a unique common fixed point, that is, there exists a unique \( x \) in \( X \) such that

\[
f(x, x) = \psi x = g(x, x) = \phi x = x.
\]

**Proof.** Let \((f, \phi)\) and \((g, \psi)\) satisfy (CLRg) property then there exist sequences \( \{x_n\}, \{y_n\}, \{x'_n\} \) and \( \{y'_n\} \) in \( X \) such that

\[
\lim_{n \to \infty} f(x_n, y_n) = \lim_{n \to \infty} \phi(x_n) = \phi \alpha \tag{3.1}
\]

\[
\lim_{n \to \infty} f(y_n, x_n) = \lim_{n \to \infty} \phi(y_n) = \phi \beta \tag{3.2}
\]

\[
\lim_{n \to \infty} g(x'_n, y'_n) = \lim_{n \to \infty} \psi(x'_n) = \psi \alpha' \tag{3.3}
\]

\[
\lim_{n \to \infty} g(y'_n, x'_n) = \lim_{n \to \infty} \psi(y'_n) = \psi \beta' \tag{3.4}
\]

for some \( \alpha, \beta, \alpha', \beta' \in X \).

Now we will show that \((f, \phi)\) and \((g, \psi)\) have common coupled coincidence point. For this, we will first show that \( \phi \alpha = \psi \alpha' \).

Putting \( x = x_n, y = y_n, u = x'_n, v = y'_n \) in condition (1) we get

\[
d(f(x_n, y_n), g(x'_n, y'_n)) \leq p \max \{d(\phi x_n, \psi x'_n), d(f(x_n, y_n), \phi x_n), d(g(x'_n, y'_n), \psi x'_n), \\
\quad d(f(x_n, y_n), \psi x'_n), d(g(x'_n, y'_n), \phi x_n) \} \]

Taking limit as \( n \to \infty \) and using (3.1), (3.2), (3.3) and (3.4), we have

\[
d(\phi \alpha, \psi \alpha') \leq p \max \{d(\phi \alpha, \psi \alpha'), d(\phi \alpha, \phi \alpha), d(\psi \alpha', \psi \alpha'), d(\phi \alpha, \psi \alpha'), d(\psi \alpha', \phi \alpha) \}
\]

\[
\Rightarrow \quad d(\phi \alpha, \psi \alpha') \leq pd(\phi \alpha, \psi \alpha')
\]

\[
\Rightarrow \quad |d(\phi \alpha, \psi \alpha')| \leq p|d(\phi \alpha, \psi \alpha')|
\]
which is possible when $\phi \alpha = \psi \alpha'$. 
So $\phi \alpha = \psi \alpha'$.
Similarly we can show that $\phi \beta = \psi \beta'$.
Now we will show that $\phi \beta = \psi \alpha'$.
For this, we put $x = y_n, y = x_n, u = x_n', v = y_n'$ in condition (1), we get

$$d(f(y_n, x_n), g(x_n', y_n')) \preceq p \max\{d(\phi y_n, \psi x_n'), d(f(y_n, x_n), \phi y_n), d(g(x_n', y_n'), \psi x_n'),$$

$$d(f(y_n, x_n), \psi x_n'), d(g(x_n', y_n'), \phi y_n]\}$$

Taking limit as $n \to \infty$ and using (3.1), (3.2), (3.3) and (3.4), we have

$$d(\phi \beta, \psi \alpha') \preceq p \max\{d(\phi \beta, \psi \alpha'), d(\phi \beta, \phi \beta), d(\psi \alpha', \psi \alpha'), d(\phi \beta, \psi \alpha'), d(\psi \alpha', \phi \beta)\}$$

$$\Rightarrow d(\phi \beta, \psi \alpha') \preceq p d(\psi \alpha', \phi \beta)$$

$$\Rightarrow |d(\phi \beta, \psi \alpha')| \leq p |d(\psi \alpha', \phi \beta)|$$

which is possible when $\phi \beta = \psi \alpha'$.
So $\phi \beta = \psi \alpha'$.
Similarly we can show that $\phi \alpha = \psi \beta'$.
Hence

$$\phi \alpha = \phi \beta = \psi \alpha' = \psi \beta'$$

(3.5)

Now we will show that $\phi \alpha = g(\alpha', \beta')$ and $\phi \beta = g(\beta', \alpha')$.
For this we put $x = x_n, y = y_n, u = \alpha', v = \beta'$ in condition (1), we get

$$d(f(x_n, y_n), g(\alpha', \beta')) \preceq p \max\{d(\phi x_n, \psi \alpha'), d(f(x_n, y_n), \phi x_n), d(g(\alpha', \beta'), \psi \alpha'),$$

$$d(f(x_n, y_n), \psi \alpha'), d(g(\alpha', \beta'), \phi x_n)\}$$

Taking limit as $n \to \infty$ and using (3.1), (3.2), (3.3), (3.4) and (3.5), we have

$$d(\phi \alpha, g(\alpha', \beta')) \preceq p \max\{d(\phi \alpha, \psi \alpha'), d(\phi \alpha, \phi \alpha), d(g(\alpha', \beta'), \psi \alpha'),$$

$$d(\phi \alpha, \psi \alpha'), d(g(\alpha', \beta'), \phi \alpha)\}$$
\[ d(\phi\alpha, g(\alpha', \beta')) \preceq p \max\{0, 0, d(g(\alpha', \beta'), \phi\alpha), 0, d(g(\alpha', \beta'), \phi\alpha)\} \]

\[ d(\phi\alpha, g(\alpha', \beta')) \preceq p d(\phi\alpha, g(\alpha', \beta')) \]

\[ |d(\phi\alpha, g(\alpha', \beta'))| \leq p |d(\phi\alpha, g(\alpha', \beta'))| \]

which is possible when as \( \phi\alpha = g(\alpha', \beta') \) as \( 0 < p < 1 \).

So \( \phi\alpha = g(\alpha', \beta') \).

Similarly \( \phi\beta = g(\beta', \alpha') \).

Now we will show that \( \psi\alpha = f(\alpha, \beta) \) and \( \psi\beta = f(\beta, \alpha) \).

For this we put \( x = \alpha, y = \beta, u = x'_n \) and \( v = y'_n \) in condition (1), we get

\[ d(f(\alpha, \beta), g(x'_n, y'_n)) \preceq p \max\{d(\phi\alpha, \psi x'_n), d(f(\alpha, \beta), \phi\alpha), d(g(x'_n, y'_n), \psi x'_n),\]

\[ d(f(\alpha, \beta), \psi x'_n), d(g(x'_n, y'_n), \phi\alpha)\} \]

Taking limit as \( n \to \infty \) and using (3.1), (3.2), (3.3), (3.4) and (3.5), we have

\[ d(f(\alpha, \beta), \psi\alpha') \preceq p \max\{d(\phi\alpha, \psi\alpha'), d(f(\alpha, \beta), \phi\alpha), d(\psi\alpha', \psi\alpha'),\]

\[ d(f(\alpha, \beta), \psi\alpha'), d(\psi\alpha', \phi\alpha)\} \]

\[ \Rightarrow d(f(\alpha, \beta), \psi\alpha') \preceq p \max\{0, d(f(\alpha, \beta), \psi\alpha'), 0, d(f(\alpha, \beta), \psi\alpha'), 0\} \]

\[ \Rightarrow d(f(\alpha, \beta), \psi\alpha') \preceq p d(f(\alpha, \beta), \psi\alpha') \]

\[ \Rightarrow |d(f(\alpha, \beta), \psi\alpha')| \leq p |d(f(\alpha, \beta), \psi\alpha')| \]

possible when \( f(\alpha, \beta) = \psi\alpha' \) as \( 0 < p < 1 \).

So \( f(\alpha, \beta) = \psi\alpha' \).

Similarly \( f(\beta, \alpha) = \psi\beta' \).

Thus \( \phi\alpha = \phi\beta = \psi\alpha' = \psi\beta' = f(\alpha, \beta) = f(\beta, \alpha) = g(\alpha', \beta') = g(\beta', \alpha') \)

\[ \Rightarrow g(\alpha', \beta') = \phi\alpha = \psi\alpha' = f(\alpha, \beta) \]

\[ \Rightarrow g(\beta', \alpha') = \phi\beta = \psi\beta' = f(\beta, \alpha). \]
Hence the pairs \((f, \phi)\) and \((g, \psi)\) have common coupled coincidence point.

Now let 
\[ f(\alpha, \beta) = \phi\alpha = g(\alpha', \beta') = \psi\alpha' = x \]
and 
\[ f(\beta, \alpha) = \phi\beta = g(\beta', \alpha') = \psi\beta' = y \, . \]

Since \((f, \phi)\) and \((g, \psi)\) are weakly compatible so

\[
\phi f(\alpha, \beta) = f(\phi\alpha, \phi\beta) = f(x, y) \quad \text{and} \quad \phi f(\beta, \alpha) = f(\phi\beta, \phi\alpha) = f(y, x),
\]
but

\[
f(\alpha, \beta) = x \Rightarrow \phi f(\alpha, \beta) = \phi x
\]

\[
f(\beta, \alpha) = y \Rightarrow \phi f(\beta, \alpha) = \phi y
\]

Therefore \(\phi x = f(x, y)\) and \(\phi y = f(y, x)\) .

Similarly \(\psi x = g(x, y)\) and \(\psi y = g(y, x)\) .

Hence

\[
\phi x = f(x, y), \phi y = f(y, x) \quad \text{and} \quad \psi x = g(x, y), \psi y = g(y, x).
\]

Now we will show that \(x = y\). Using condition (1), we get

\[
d(x, y) = d(f(\alpha, \beta), g(\beta', \alpha'))
\]

\[
\preceq p \max \{d(\phi\alpha, \psi\beta''), d(f(\alpha, \beta), \phi\alpha), d(g(\beta', \alpha'), \psi\beta'),
\]

\[
d(f(\alpha, \beta), \psi\beta'), d(g(\beta', \alpha'), \phi\alpha)\}\}
\]

\[
\Rightarrow d(x, y) \preceq p \max \{0, 0, 0, 0\}
\]

\[
\Rightarrow |d(x, y)| = 0
\]

\[
\Rightarrow x = y
\]

Now, we will prove that \(\phi x = \psi x\) .

Using condition (1), we have

\[
d(\phi x, \psi x) = d(f(x, y), g(x, y))
\]

\[
\preceq p \max \{d(\phi x, \psi x), d(f(x, y), \phi x), d(g(x, y), \psi x)\},
\]
Savitri Hooda, N

\[ d(f(x, y), \psi x), d(g(x, y), \phi x) \]

\[ \Rightarrow \]

\[ d(\phi x, \psi x) \preceq p \max \{d(\phi x, \psi x), 0, 0, d(\phi x, \psi x), d(\psi x, \phi x)\} \]

\[ \Rightarrow \]

\[ |d(\phi x, \psi x)| \leq p |d(\phi x, \psi x)| < |d(\phi x, \psi x)| \]

which is possible when \( \phi x = \psi x \) as \( 0 < p < 1 \).

So \( \phi x = \psi x \).

\[ \Rightarrow \]

\[ f(x, y) = \phi x = \psi x = g(x, y) \]

Similarly \( \phi y = \psi y \) and \( f(y, x) = g(y, x) \).

Now we will show that \( \phi x = x \).

Using condition (1), we get

\[ d(x, \phi x) = d(f(\alpha, \beta), g(x, y)) \]

\[ \preceq p \max \{d(\phi \alpha, \psi x), d(f(\alpha, \beta), \phi \alpha), d(g(x, y), \psi x), d(f(\alpha, \beta), \psi x), d(g(x, y), \phi \alpha)\} \]

\[ \Rightarrow \]

\[ d(x, \phi x) \preceq p \max \{d(x, \psi x), d(f(\alpha, \beta), \phi \alpha), d(\phi x, \psi x), d(f(\alpha, \beta), \psi x), d(g(x, y), \phi \alpha)\} \]

\[ \Rightarrow \]

\[ d(x, \phi x) \preceq p \max \{d(x, \phi x), d(x, x), d(\phi x, \phi x), d(\psi x, x), d(\phi x, x)\} \]

\[ \Rightarrow \]

\[ d(x, \phi x) \preceq p \max \{d(x, \phi x), 0, 0, d(\phi x, x), d(\phi x, x)\} \]

\[ \Rightarrow \]

\[ |d(x, \phi x)| \leq p \max |d(x, \phi x)| \]

which is possible when \( x = \phi x \) as \( 0 < p < 1 \).

So \( x = \phi x \).

Hence \( f(x, x) = \psi x = g(x, x) = \phi x = x \).

Thus \( f, g, \phi \) and \( \psi \) have a common fixed point.

Now to prove uniqueness, let \( y \) be any other common fixed point of \( f, g, \phi \) and \( \psi \).

\[ \Rightarrow \]

\[ f(y, y) = \psi y = g(y, y) = \phi y = y \]
Then \( d(x, y) = d(f(x, x), g(y, y)) \)
\[
\lesssim p \max \{d(\phi x, \psi y), d(f(x, x), \phi x), d(g(y, y), \psi y), \}
\]
\[
d(f(x, x), \psi y), d(g(y, y), \phi x)\}
\]
\[
\Rightarrow d(x, y) \lesssim p \max \{d(x, y), d(x, x), d(y, y), d(x, y), d(x, x)\}
\]
\[
\Rightarrow |d(x, y)| \leq p |d(x, y)|
\]
which is possible when \( x = y \) as \( 0 < p < 1 \).
So \( x = y \).
Hence \( f, g, \phi \) and \( \psi \) have unique common fixed point.

**Example 3.1.** Let \( X = R \) be a complex valued metric space equipped with the complex valued metric space \( d(x, y) = |x - y| i \).
Let \( f: X \times X \to X \) and \( g: X \times X \to X \) be defined for all \( x, y \in X \) as
\[
f(x, y) = \begin{cases} 
\frac{x - y}{8} & \text{if } x \geq y \\
0 & \text{if } x < y 
\end{cases},
\]
\[
g(x, y) = \begin{cases} 
\frac{x - y}{10} & \text{if } x \geq y \\
0 & \text{if } x < y 
\end{cases}
\]
Let \( \psi: X \to X \) and \( \phi: X \to X \) be defined as
\[
\psi(x) = \frac{x}{2}, \ldots \phi(x) = \frac{x}{30}, \ldots, \text{ for all } x \in X.
\]
It is easy to check that all conditions of Theorem 3.1 are satisfied for all \( x, y, u, v \in X \). Thus, we have \( x = 0 \) is the unique common fixed point of \( f, g, \phi \) and \( \psi \).
If \( g = f \) and \( \psi = \phi \) in Theorem 3.1 then we have the following corollary:

**Corollary 3.1.** Let \((X, d)\) be a complex-valued metric-space and let \( f: X^2 \to X \) and \( \phi: X \to X \) are mappings such that
\[
(1) \quad d(f(x, y), f(u, v)) \leq p \max \{d(\phi x, \phi u), d(f(x, y), \phi x), d(g(u, v), \phi u),
\]
\[
d(f(x, y), \phi u), d(f(u, v), \phi x)\}
\]
for all \( x, y, u, v \in X \) and \( 0 < p < 1 \),
(2) the pair \((f, \phi)\) is weakly compatible.

If the pair \((f, \phi)\) satisfy \((\text{CLRg})\) property then there exists a unique \(x\) in \(X\) such that \(f(x, x) = \phi x = x\).

ACKNOWLEDGEMENT

The authors sincerely thank the referees for their careful reading and valuable suggestions which have improved this paper.

REFERENCES


Common Fixed Point Theorem
For Mappings Satisfying (CLRg) Property


[13] Lakshmikantham, V., Ciric, L., Coupled fixed point theorems for non linear contractions in partially ordered metric spaces.


