Absolute Mean Graceful Labeling in Path Union of Various Graphs

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1. Introduction

Throughout present paper, we shall acknowledge $G = (p, q)$, a finite, simple and undirected graph with $V(G)$-vertex set having $p$ vertices and $E(G)$-edge set having $q$ edges. For a graph $G = (V, E)$, a function with domain $V$ or $E$ or $V \cup E$ is known as a graph labeling for $G$. Graceful labeling of a graph $G$ is popular concept firstly established by Alexander (Rosa 1967). The name graceful labeling was given by (Golomb 1972) which was earlier familiar as $\beta$-valuation. Kaneria, Makadia and Meghapara (Kaneria 2015) proved graceful labeling for grid related graph. Kaneria and Makadia (Kaneria 2015) proved graceful labeling for double step grid graph. All path graphs $P_1$, cycle $C_2$, and complete bipartite graph $K_{m,n}$ were proved graceful graphs in the early researches in study of graceful labeling. Kaneria and Chudasama (Kaneria 2017) introduced absolute mean graceful labeling and proved that it holds true for this new labeling. Current paper is to study the same labeling for path union of finite number of copies of above mentioned graphs and enhances wide scope of operations on such graphs consisting absolute mean graceful labeling. For comprehensive learning of graph labeling, we referred Gallian (Gallian 2011).

Take path $P_1, P_2, P_{m-1}, ..., P_q, P_t$ and put them vertically. A graph made by joining horizontal vertices of paths $P_1, P_{m-1}, ..., P_q, P_t$ is defined as step grid graph and denoted by $St_t$, $n \geq 3$. It is obvious that $|V(St_t)| = \frac{1}{2}(n^2 + 3n - 2)$ and $|E(St_t)| = n + n - 2$. Similarly, by arranging $P_2, P_1, ..., P_q, P_{m-1}, ..., P_q, P_t$ and then joining vertices horizontally, we get double step grid graph $DSt_t$. A function $f$ is said to be an absolute mean graceful labeling of a graph $G$, if $f: V(G) \rightarrow \{0, \pm 1, \pm 2, ..., \pm |E|\}$ is injective and edge labeling function $f^*$ defined as $f^*(e) = \frac{|f(u) - f(v)|}{2}$ is bijective, $\forall e = (u, v) \in E(G)$. A graph which admits such labeling is called absolute mean graceful graph. In the previous work of this labeling, many graphs has been proved as absolute mean graceful graphs such as all path graph $P_1$, cycle graph $C_2$, complete bipartite graph $K_{m,n}$, grid graph $P_n \times P_n$, step grid graph $St_t$ and double step grid graph $DSt_t$. For a graph $G$, if $G_1, G_2, ..., G_t (t \geq 2)$ are $t$ copies of $G$, then graph made by adding an edge from $G_i$ to $G_{i+1}$ (for $i = 1, 2, ..., t-1$) is said to be path union of $G$ which is denoted as $P(G_1, G_2, ..., G_t)$ or $P(t.G)$.

2. Main Results

Theorem 1: Every $P(t \cdot P_t)$ is absolute mean graceful graph.

Proof: Let $P_t$ be absolute mean graceful graph having $p$ and $q$ numbers of vertices and edges respectively. So that $p = n$ and $q = n - 1$.

Since $P_t$ is absolute mean graceful graph proved by Kaneria and Chudasama (Kaneria 2017), there exists
absolute mean graceful labeling $f: V(P) \to \{0, \pm 1, \pm 2, \ldots, \pm |E|\}$ and edge labeling function $f^*: E(P) \to \{1, 2, \ldots, |E|\}$ defined as $f^*(e) = \frac{|f(u) - f(v)|}{2}$, which are injective and bijective respectively, $\forall e = (u, v) \in E(P)$. Let $v_{1, 1}, v_{1, 2}, \ldots, v_{1, s}$ be vertices of $P(t \cdot P)_{\ast}$ made up with $t$ copies of path graph $P_{\ast}$ by joining $v_{1, 1}$ and $v_{1, 1, 1}$.

$g(v_{1, j}) = \begin{cases} f(v_{1, j}), & \forall i = 2, 3, \ldots, t \\ (-1)^{i-1} |g(v_{1, i-1})| + q + 1, & \forall j = 1, 2, \ldots, n. \end{cases}$

Case II: When $n$ is an odd number

$g(v_{1, j}) = \begin{cases} f(v_{1, j}), & \forall i = 2, 3, \ldots, t \\ (-1)^{i-1} |g(v_{1, i-1})| + q + 1, & \forall j = 1, 2, \ldots, n. \end{cases}$

Which is an injective vertex labeling function for $P(t \cdot C)$. It is easy to check that edge labeling function $g^*$ is bijective. Therefore, $P(t \cdot C)_{\ast}$ is an absolute mean graceful graph.

Illustration 1. Absolute mean graceful labeling in path union of 4 copies of cycle $C_n$.

Theorem III: Every $P(t \cdot K_{m,n})$ is absolute mean graceful graph.

Proof: Let $K_{m,n}$ be absolute mean graceful graph with $p$ and $q$ number of vertices and edges. So that $p = m + n$ and $q = mn$.

Let $V(C) = \{u_1, u_2, \ldots, u_s\}. Let V(P(t \cdot C)) = \{u_1, u_2, \ldots, u_s\}$ be vertex set of path union graph $G$ made up with $t$ copies of cycle graph $C_s$ by joining $u_{1, i}$ and $u_{1, i+1}$. Where $q = \frac{n+2}{2}$, $\forall i = 1, 2, \ldots, t – 1$, where $u_2$ denotes $p$th vertex of $g^*$ copy of $C_s$ in $G$, $\forall i = 1, 2, \ldots, t$ and $\forall j = 1, 2, \ldots, n$. So that $P^* = \{|V(P(t \cdot C))| = mt\}$ and $Q^* = |E(P(t \cdot C))| = t(n + 1) – 1$.

Let us define vertex labeling function $g: V(P(t \cdot P_{\ast})) \to \{0, \pm 1, \pm 2, \ldots, \pm Q^*\}$ as:

$\forall i = 1, 2, \ldots, t$ and $j = 1$

$\forall i = 1, 2, \ldots, t$; $\forall j = 2, 3, \ldots, n$.

Since $K_{m,n}$ is absolute mean graceful graph proved by Kaneria and Chudasama [5], there exists absolute mean graceful labeling $f: V(K_{m,n}) \to \{0, \pm 1, \pm 2, \ldots, \pm q\}$ and an edge labeling function $f^*: E(K_{m,n}) \to \{1, 2, \ldots, |E|\}$ which are injective and bijective respectively.

Let $V(K_{m,n}) = \{u_{i, 1}, u_{i, 2}, \ldots, u_{i, n}\}$ be vertex set of path union graph $P(t \cdot K_{m,n})$ made up with $t$ copies of complete bipartite graph $K_{m,n}$ by joining

(1) $i_{1}$, where $k = \left\lfloor \frac{m+1}{2} \right\rfloor$ and vertex $u_{i, 1}$, if $m$ is an even number;

(2) vertex $u_{i, 1}$, where $k = \left\lfloor \frac{m+1}{2} \right\rfloor$ and vertex $u_{i, 1}$, if $m$ is an odd number. Which holds for $\forall i = 1, 2, \ldots, t – 1$. 

Theorem II: Every $P(t \cdot C)_{\ast}$ for $n$ - an even number is absolute mean graceful graph.

Proof: Let $C_n$ for $n$ - an even number is absolute mean graceful graph having $p$ and $q$ number of vertices and edges respectively. So that $p = n$ and $q = n$.

Let $C_n$ be an absolute mean graceful graph by Kaneria and Chudasama [5], let $f: V(C) \to \{0, \pm 1, \pm 2, \ldots, \pm |E|\}$ and $f^*: E(C) \to \{1, 2, \ldots, |E|\}$ be vertex and edge labelings.

Let $V(C) = \{u_1, u_2, \ldots, u_s\}$. Let $V(P(t \cdot C)) = \{u_1, u_2, \ldots, u_s\}$ be vertex set of path union graph $G$ made up with $t$ copies of cycle graph $C_s$ by joining $u_{1, i}$ and $u_{1, i+1}$ where $k = \frac{n+2}{2}$, $\forall i = 1, 2, \ldots, t – 1$, where $u_2$ denotes $p$th vertex of $g^*$ copy of $C_s$ in $G$, $\forall i = 1, 2, \ldots, t$ and $\forall j = 1, 2, \ldots, n$. So that $P^* = |V(P(t \cdot C))| = mt$ and $Q^* = |E(P(t \cdot C))| = t(n + 1) – 1$.

Let us define vertex labeling function $g: V(G) \to \{0, \pm 1, \pm 2, \ldots, \pm Q^*\}$ as:

Case I: When $\frac{n}{2}$ is an even number

$\forall i = 1$ and $\forall j = 1, 2, \ldots, n$

$\forall i = 3, 5, 7, \ldots, t – 1$ or $t$

$\forall j = 2, 3, \ldots, n$

$\forall i = 2, 4, 6, \ldots, t – 1$ or $t$

$\forall j = 2, 3, \ldots, n.$

Since $K_{m,n}$ is absolute mean graceful graph proved by Kaneria and Chudasama [5], there exists absolute mean graceful labeling $f: V(K_{m,n}) \to \{0, \pm 1, \pm 2, \ldots, \pm q\}$ and an edge labeling function $f^*: E(K_{m,n}) \to \{1, 2, \ldots, |E|\}$ which are injective and bijective respectively.

Let $V(K_{m,n}) = \{u_1, u_2, \ldots, u_s\}$ be vertex set of path union graph $P(t \cdot K_{m,n})$ made up with $t$ copies of complete bipartite graph $K_{m,n}$ by joining

(1) vertex $u_{i, k}$, where $k = \left\lfloor \frac{m+1}{2} \right\rfloor$ and vertex $u_{i, 1}$, if $m$ is an even number;

(2) vertex $u_{i, 1}$, where $k = \left\lfloor \frac{m+1}{2} \right\rfloor$ and vertex $u_{i, 1}$, if $m$ is an odd number. Which holds for $\forall i = 1, 2, \ldots, t – 1$. 

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It is obvious that $u_i, j$ and $u_i, k$ denotes $j$th and $k$th vertices of $M$ and $N$ parts in $i$th copy of $K_m, n$ in $P(t \cdot K_m, n)$, $\forall \ i = 1, 2, ..., t; \forall \ j = 1, 2, ..., m$ and $\forall \ k = 1, 2, ..., n$. Clearly, $P' = |V(P(t \cdot K_m, n))| = t(m + n)$ and $Q' = |E(P(t \cdot K_m, n))| = tmn + t - 1$. Let us define vertex labeling function $g: V(P(t \cdot K_m, n)) \to \{0, \pm 1, \pm 2, ..., \pm Q'\}$ as:

$$
 g(u_{i,j}) = \begin{cases} 
 f(u_i), & i = 1 \text{ and } \forall \ j = 1, 2, ..., m \\
 g(u_{i-1,j}) + q + 1, & \forall \ i = 2, 3, ..., t, \forall \ j = 1, 2, ..., m.
\end{cases}
$$

$$
 g(v_{i,j}) = \begin{cases} 
 f(v_i), & i = 1 \text{ and } \forall \ k = 1, 2, ..., n \\
 g(v_{i-1,j}) - q - 1, & \forall \ i = 2, 3, ..., t, \forall \ k = 1, 2, ..., n.
\end{cases}
$$

Which gives an injective vertex labeling of $P(t \cdot K_m, n)$. It is easy to verify that an edge labeling function $g^*$ is bijective function. Therefore, $P(t \cdot K_m, n)$ is an absolute mean graceful graph.

**Illustration 2:** Absolute mean graceful labeling in 5 copies of $K_{3,4}$.

**Theorem IV:** Every $P(t \cdot P_m \times P_n)$ is absolute mean graceful graph.

**Proof:** Let $P_m \times P_n$ be absolute mean graceful graph with $p$ and $q$ number of vertices and edges respectively. So that $p = mn$ and $q = 2mn - m - n$.

Since $P_m \times P_n$ is absolute mean graceful graph proved by Kaneria and Chudasama [5], let $f^*: V(P_m \times P_n) \to \{0, \pm 1, \pm 2, ..., \pm q\}$ and $f^*: E(P_m \times P_n) \to \{1, 2, ..., q\}$ be vertex labeling injective and edge labeling bijective functions respectively.

Let $u_{i,k}$ be vertices of $P_m \times P_n$, $\forall \ i = 1, 2, ..., m, \forall \ k = 1, 2, ..., n$. Let $u_{i,k}$ be the $u_{i,k}$ located vertex of $i$th copy of $P_m \times P_n$ of $P(t \cdot P_m \times P_n)$ which is made up with $t$ copies of grid graph $P_m \times P_n$ by joining

(1) vertex $u_{i,1}$ and vertex $u_{i+1,m,n}$ if $q$ is an odd number or
(2) vertex $u_{i,1}$ and vertex $u_{i+1,m,n}$ if $q$ is an even number,
which holds true for $\forall i = 1, 2, ..., t - 1; \forall j = 1, 2, ..., m; \forall k = 1, 2, ..., n$. It is obvious that $P' = |V(P(t \times P_m \times P))| = tmn$ and $Q' = |E(P(t \times P_m \times P))| = t[2mn - m - n + 1] - 1$.

Proof: Union of 3 copies of step grid graph $Kaneria$ and Chudasama (Kaneria 2017), let labeling and hence it is absolute mean graceful graph.

Since $DSt_2$ is absolute mean graceful graph proved by Kaneria and Chudasama [5], let $f: V(DSt_2) \rightarrow \{0, \pm 1, \pm 2, ..., \pm q\}$ be injective vertex labeling and bijective edge labeling functions respectively. From the definition of absolute mean graceful labeling which is injective. Therefore, $P(t \times P_m \times P)$ is an absolute mean graceful graph.

Theorem V: Every $P(t \cdot St_r)$ is absolute mean graceful graph.

Proof: Let $St_r$ be absolute mean graceful graph with $p$ and $q$ number of vertices and edges respectively. So that $p = \frac{n^2 + 3n - 2}{2}$ and $q = n_1 + n - 2$.

Let $u_{i,j,k}(1 \leq k \leq n)$ be vertices in first row, $u_{i,j,k}(1 \leq k \leq n)$ be vertices in second row, $u_{i,j,k}(2 \leq k \leq n)$ be vertices in third row, $u_{i,j,k}(j - 1 \leq k \leq n)$ be vertices in $j^{th}$ row and $u_{i,j,k}(n - 1 \leq k \leq n)$ be vertices in $n^{th}$ row. Let $u_{i,j,k}$ be $u_{i,j,k}$ located vertex of $j^{th}$ copy of $St_r$ of $G$ which is made up with $t$ copies of $St_r$ by joining vertices $u_{i,j,k}$ and $u_{i+1,j,k}$; $\forall j = 1, 2, ..., m; \forall k = 1, 2, ..., n$. So that $P' = |V(G)| = \frac{t}{n^2 + 3n - 2}$ and $Q' = |E(G)| = t(n^2 + n - 1) - 1$.

Define vertex labeling $g: V(P(t \times P_m \times P)) \rightarrow \{0, \pm 1, \pm 2, ...\}$ and $m_1 \leq m \leq m_2$.

Let $u_{i,j,k}$ be $u_{i,j,k}$ located vertex of $j^{th}$ copy of $DSn_2$ of $P(t \times St_r)$ which is made up with $t$ copies of $DSn_2$ by joining vertices $u_{i,j,k}$ and $u_{i+1,j,k}$, where $l = \frac{n + 2}{2}$ and $\forall i = 1, 2, ..., l$. It is obvious that $P' = |V(P(t \times St_r))| = \frac{tm}{4}(n + 6)$ and $Q' = |E(P(t \times St_r))| = \frac{t}{4}\left\{\frac{n^2 + 3n}{2}\right\} - 1$.

Let us define vertex labeling function $g: V(P(t \times St_r)) \rightarrow \{0, \pm 1, \pm 2, ..., \pm Q'\}$ and $\forall i = 1, 2, ..., n$.

Illustration 3. Absolute mean graceful labeling in path union of 3 copies of step grid graph $St_r$.

Theorem VI: Every $P(t \cdot DSt_r)$ is absolute mean graceful graph.

Proof: Let $DSt_r$ be absolute mean graceful graph with $p$ and $q$ number of vertices and edges respectively. So that $p = \frac{n}{4}(n + 6)$ and $q = \frac{n^2 + 3n - 2}{2}$.

Since $DSt_2$ is absolute mean graceful graph proved by Kaneria and Chudasama (Kaneria 2017), let $f: V(DSt_2) \rightarrow \{0, \pm 1, \pm 2, ..., \pm q\}$ and $f^*: E(DSt_2) \rightarrow \{1, 2, ..., q\}$ be injective vertex labeling and bijective edge labeling functions respectively.
Fig 3 Absolute mean graceful labeling for path union of 3 copies of step grid graph $S_n$ with $|V(T)| = 13$ and $|E(G)| = 18$

$$g(v_{i,j,k}) = \begin{cases} 
    f(v_{j,k}), & i = 1 \\
    g(v_{i-1,j,k}) - q - 1, & \text{if } g(v_{i-1,j,k}) \leq -2, \ \forall i = 2, 3, \ldots, t; \\
                       -q - 1, & \text{if } g(v_{i-1,j,k}) = 0, \ \forall i = 2, 3, \ldots, t; \\
                       q + 1, & \text{if } g(v_{i-1,j,k}) = -1, \ \forall i = 2, 3, \ldots, t; \\
    g(v_{i-1,j,k}) + q + 1, & \text{if } g(v_{i-1,j,k}) \geq 2, \ \forall i = 2, 3, \ldots, t; 
\end{cases}$$

Which is an injective function for vertex labeling of $P(t \times T)$. It is clear that induced edge labeling function $g^*$ can be defined as per definition of absolute mean graceful labeling which is bijective function. Therefore, $P(t \times T)$ holds absolute mean graceful labeling and hence it is absolute mean graceful graph.

**Theorem VII**: Every $P(t \times T)$ is absolute mean graceful graph, where $T$ consists absolute mean graceful labeling.

**Proof**: Let $T$ be absolute mean graceful tree with $p = |V(T)|$ and $q = |E(T)|$. It is clear that $q = p - 1$.

Since $T$ is absolute mean graceful tree, there exists absolute mean graceful labeling $f : V(T) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm q\}$ which is an injective function and the induced edge labeling function $f^* : E(T) \rightarrow \{1, 2, \ldots, q\}$ defined as

$$f^*(e) = \left\lfloor \frac{|f(u) - f(v)|}{2} \right\rfloor$$

is bijective for every edge $e = (u, v) \in E(T)$.

Let $V(T) = \{u_1, u_2, \ldots, u_p\}$. Let $u_{i, 0}, u_{i, 1}, \ldots, u_{i, p}$ be vertices of path union graph $P(t \times T)$ made up with $t$ copies of tree graph $T$ by joining $u_{i, 0}$ and $u_{i+1, 0}$, for $i = 1, 2, \ldots, t - 1$ and $\forall j = 1, 2, \ldots, p$, where $u_{i,j}$ denotes $j$th vertex of $i$th copy of $T$ in $P(t \times T)$. So that $P' = |V(P(t \times T))| = tp$ and $Q' = |E(P(t \times T))| = tp - 1$. By defining vertex labeling function $g : V(P(t \times T)) \rightarrow \{0, \pm 1, \pm 2, \ldots, \pm Q\}$ as:

$$g(v_{i,j}) = \begin{cases} 
    f(v_j), & i = 1 \text{ and } \forall j = 1, 2, \ldots, p \\
    g(v_{i-1,j}) - q - 1, & \text{if } g(v_{i-1,j}) \leq 0, \ \forall i = 2, 3, \ldots, t; \\
                       q + 1, & \text{if } g(v_{i-1,j}) > 0, \ \forall i = 2, 3, \ldots, t; \\
                       g(v_{i-1,j}) + q + 1, & \text{if } g(v_{i-1,j}) \geq 2, \ \forall i = 2, 3, \ldots, t; \\
                       g(v_{i-1,j}) - q - 1, & \text{if } g(v_{i-1,j}) > 0, \ \forall i = 2, 3, \ldots, t; \\
\end{cases}$$

Which is an injective function for vertex labeling of $P(t \times T)$. It is clear that induced edge labeling function $g^*$ can be defined as per definition of absolute mean graceful labeling which is bijective. Hence, $P(t \times T)$ holds absolute mean graceful labeling it is absolute mean graceful graph.

**Illustration 4**: Absolute mean graceful labeling in path union of 3 copies of tree $T$. 

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Fig 4 Absolute mean graceful labeling for path union of 3 copies of tree $T$ with $|V(T)| = 17$ and $|E(G)| = 16$

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