Fibonacci and $k$ Lucas Sequences as Series of Fractions

A. D. GODASE$^1$ AND M. B. DHAKNE$^2$

$^1$V. P. College, Vaijapur, Maharashtra, India
$^2$Dr. B. A. M. University, Aurangabad, Maharashtra, India

E-mail: mathematicsdept@vpcollege.net

Received: July 29, 2015 | Revised: September 30, 2015 | Accepted: October 31, 2015
Published online: March 30, 2016
The Author(s) 2016. This article is published with open access at www.chitkara.edu.in/publications

Abstract In this paper, we defined new relationship between $k$ Fibonacci and $k$ Lucas sequences using continued fractions and series of fractions, this approach is different and never tried in $k$ Fibonacci sequence literature.

Keywords: $k$-Fibonacci sequence, $k$-Lucas sequence, Recurrence relation

Mathematics Subject Classification: 11B39, 11B83

1 INTRODUCTION

The Fibonacci sequence is a source of many nice and interesting identities. Many identities have been documented in [2], [3], [10], [11], [12], [13], [17]. A similar interpretation exists for $k$ Fibonacci and $k$ Lucas numbers. Many of these identities have been documented in the work of Falcon and Plaza[1], [4], [5], [7], [8], [9], where they are proved by algebraic means. In this paper, we obtained some new properties for $k$ Fibonacci and $k$ Lucas sequences using series of fraction.

2 PRELIMINARY

Definition 2.1. The $k$–Fibonacci sequence $\{F_{k,n}\}_{n=1}^{\infty}$ is defined as, $F_{k,n+1} = k \cdot F_{k,n} + F_{k,n-1}$, with $F_{k,0} = 0$, $F_{k,1} = 1$, for $n \geq 1$
Definition 2.2. The $k-$ Lucas sequence $\{L_{k,n}\}_{n=1}^{\infty}$ is defined as, $L_{k,n+1} = k \cdot L_{k,n} + L_{k,n-1}$, with $L_{k,0} = 2, L_{k,1} = k$, for $n \geq 1$

Characteristic equation of the initial recurrence relation is,

$$r^2 - k \cdot r - 1 = 0 \quad (1)$$

Characteristic roots are

$$r_1 = \frac{k + \sqrt{k^2 + 4}}{2} \quad (2)$$

and

$$r_2 = \frac{k - \sqrt{k^2 + 4}}{2} \quad (3)$$

Characteristic roots verify the properties

$$r_1 - r_2 = \sqrt{k^2 + 4} = \sqrt{\Delta} = \delta \quad (4)$$

$$r_1 + r_2 = k \quad (5)$$

$$r_1 \cdot r_2 = -1 \quad (6)$$

Binet forms for $F_{k,n}$ and $L_{k,n}$ are

$$F_{k,n} = \frac{r_1^n - r_2^n}{r_1 - r_2} \quad (7)$$

and

$$L_{k,n} = r_1^n + r_2^n \quad (8)$$
2.1 The first few members of this $k$ Fibonacci family are

\[
\begin{align*}
1, \\
k, \\
1 + k^2, \\
2k + k^3, \\
1 + 3k^2 + k^4, \\
3k + 4k^3 + k^5, \\
1 + 6k^2 + 5k^4 + k^6, \\
4k + 10k^3 + 6k^5 + k^7, \\
1 + 10k^2 + 15k^4 + 7k^6 + k^8, \\
5k + 20k^3 + 21k^5 + 8k^7 + k^9, \\
1 + 15k^2 + 35k^4 + 28k^6 + 9k^8 + k^{10}, \\
6k + 35k^3 + 56k^5 + 36k^7 + 10k^9 + k^{11}, \\
1 + 21k^2 + 70k^4 + 84k^6 + 45k^8 + 11k^{10} + k^{12}, \\
7k + 56k^3 + 126k^5 + 120k^7 + 55k^9 + 12k^{11} + k^{13}, \\
1 + 28k^2 + 126k^4 + 210k^6 + 165k^8 + 66k^{10} + 13k^{12} + k^{14}, \\
8k + 84k^3 + 252k^5 + 330k^7 + 220k^9 + 78k^{11} + 14k^{13} + k^{15}, \\
1 + 36k^2 + 210k^4 + 462k^6 + 495k^8 + 286k^{10} + 91k^{12} + 15k^{14} + k^{16}, \\
9k + 120k^3 + 462k^5 + 792k^7 + 715k^9 + 364k^{11} + 105k^{13} + 16k^{15} + k^{17}, \\
1 + 45k^2 + 330k^4 + 924k^6 + 1287k^8 + 1001k^{10} + 455k^{12} + 120k^{14} + 17k^{16} + k^{18}, \\
10k + 165k^3 + 792k^5 + 1716k^7 + 2002k^9 + 1365k^{11} + 560k^{13} + 136k^{15} + 18k^{17} + k^{19}
\end{align*}
\]
2.2 \( k \) Fibonacci sequences in Encyclopaedia of Integer Sequences

<table>
<thead>
<tr>
<th>( F_{k,n} )</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_{1,n} )</td>
<td>A000045</td>
</tr>
<tr>
<td>( F_{2,n} )</td>
<td>A000129</td>
</tr>
<tr>
<td>( F_{3,n} )</td>
<td>A006190</td>
</tr>
<tr>
<td>( F_{4,n} )</td>
<td>A001076</td>
</tr>
<tr>
<td>( F_{5,n} )</td>
<td>A052918</td>
</tr>
<tr>
<td>( F_{6,n} )</td>
<td>A005668</td>
</tr>
<tr>
<td>( F_{7,n} )</td>
<td>A054413</td>
</tr>
<tr>
<td>( F_{8,n} )</td>
<td>A041025</td>
</tr>
<tr>
<td>( F_{9,n} )</td>
<td>A099371</td>
</tr>
<tr>
<td>( F_{10,n} )</td>
<td>A041041</td>
</tr>
<tr>
<td>( F_{11,n} )</td>
<td>A049666</td>
</tr>
</tbody>
</table>

3 \( k \) RELATIONSHIP OF THE SEQUENCES \( F_{k,N} \) AND \( L_{k,N} \) AS CONTINUED FRACTIONS:

In general, a (simple) continued fraction is an expression of the form

\[
[a_0, a_1, \ldots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}
\]  

(9)

The letters \( a_1, a_2, \ldots \) denote positive integers. The letter \( a_0 \) denotes an integer.

The expansion \( \frac{F_{k,n+1}}{F_{k,n}} \) in continued fraction is written as

\[
\frac{F_{k,n+1}}{F_{k,n}} = k + \cfrac{1}{k + \cfrac{1}{k + \cfrac{1}{k + \ddots + \cfrac{1}{k + \cfrac{1}{a_n}}}}}
\]  

(10)

Here \( n \) denotes the number of quantities equal to \( k \).

We knew that

\[
F_{k,n}^2 - F_{k,n-1}F_{k,n+1} = (-1)^{n-1}
\]  

(11)
Moreover, in general we have

\[ \frac{F_{k,n+1}}{F_{k,n}} = r_1 \frac{1 - (\frac{r_2}{r_1})^{n+1}}{1 - (\frac{r_2}{r_1})^n} \]

Let, \( r_1 \) denote the larger of the root, we have

\[ \lim_{n \to \infty} \frac{F_{k,n+1}}{F_{k,n}} = r_1 \]

More generaly formula (9) is written as

\[ \frac{F_{k,(n+1)t}}{F_{k,nt}} = L_{k,t} - \frac{(-1)^y}{L_{k,j} - \frac{(-1)^y}{L_{k,j} - ...}} \]

(12)

Here, \( n \) denotes the number of \( L_{k,t} \)'s

When \( n \) increases indefinitely, we have

\[ \lim_{n \to \infty} \frac{F_{k,(n+1)t}}{F_{k,nt}} = (r_1)^y \]

As equation (9), we have relation for \( L_{k,n} \)

\[ \frac{L_{k,(n)t}}{F_{k,(n-1)t}} = L_{k,t} - \frac{(-1)^y}{L_{k,j} - \frac{(-1)^y}{L_{k,j} - ...}} \]

(13)

Here \( n \) denotes the number of quantities equal to \( L_{k,t} \).

We knew that

\[ L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (-1)^n \Delta \]

(14)

More generally, equations (10) and (13) are modified as

\[ F_{k,nt}^2 - F_{k,(n-1)t}F_{k,(n+1)t} = (-1)^{(n-1)y}(F_{k,t})^2 \]

(15)
Moreover using (7) and (8), gives

$$\Delta F_{k,n}^2 = r_1^{2n+2t} + r_2^{2n+2t} - 2(-1)^{n+t}$$  \hspace{1cm} (17)$$

$$\Delta L_{k,n}^2 = r_1^{2n} + r_2^{2n} - 2(-1)^n$$  \hspace{1cm} (18)$$

Again by subtracting (16) and (17), gives

$$\Delta (F_{k,n+t}^2 - (-1)^t F_{k,n}^2) = (r_1^{2n+t} - r_2^{2n+t})(r_1' + r_2')$$  \hspace{1cm} (19)$$

and

$$F_{k,n+t}^2 - (-1)^t F_{k,n}^2 = \Delta F_{k,t} F_{k,2n+t}$$  \hspace{1cm} (20)$$

Similarly, we obtain

$$L_{k,n+t}^2 - (-1)^t L_{k,n}^2 = \Delta F_{k,t} L_{k,2n+t}$$  \hspace{1cm} (21)$$

4 SEQUENCES $F_{K,N}$ AND $L_{K,N}$ AS A SERIES OF FRACTIONS:

Theorem 4.1. For $n, k > 0$,

1. \[
\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} - \frac{(-1)^2}{F_{k,1} F_{k,2}} - \frac{(-1)^3}{F_{k,2} F_{k,3}} - \frac{(-1)^4}{F_{k,3} F_{k,4}} - \cdots - \frac{(-1)^{n-1}}{F_{k,n-1} F_{k,n}}
\]

2. \[
\frac{L_{k,n+1}}{L_{k,n}} = \frac{L_{k,2}}{L_{k,1}} - \frac{(-1)^2 \Delta}{L_{k,2} L_{k,1}} - \frac{(-1)^3 \Delta}{L_{k,2} L_{k,3}} - \frac{(-1)^4 \Delta}{L_{k,3} L_{k,4}} - \cdots - \frac{(-1)^n \Delta}{L_{k,n-1} L_{k,n}}
\]
**Proof.** We can write expressions of \( \frac{F_{k,n+1}}{F_{k,n}} \) and \( \frac{L_{k,n+1}}{L_{k,n}} \) in series as

\[
\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} + \left( \frac{F_{k,3}}{F_{k,2}} - \frac{F_{k,2}}{F_{k,1}} \right) + \left( \frac{F_{k,4}}{F_{k,3}} - \frac{F_{k,3}}{F_{k,2}} \right) + \cdots \\
+ \left( \frac{F_{k,n+1}}{F_{k,n}} - \frac{F_{k,n}}{F_{k,n-1}} \right)
\]

\[
= \frac{F_{k,2}}{F_{k,1}} - \frac{(F_{k,2}^2 - F_{k,3}F_{k,1})}{F_{k,1}F_{k,2}} - \frac{(F_{k,3}^2 - F_{k,2}F_{k,4})}{F_{k,2}F_{k,3}} - \cdots \\
- \frac{(F_{k,n}^2 - F_{k,n+1}F_{k,n-1})}{F_{k,n-1}F_{k,n}}
\]

And

\[
\frac{L_{k,n+1}}{L_{k,n}} = \frac{L_{k,2}}{L_{k,1}} + \left( \frac{L_{k,3}}{L_{k,2}} - \frac{L_{k,2}}{L_{k,1}} \right) + \left( \frac{L_{k,4}}{L_{k,3}} - \frac{L_{k,3}}{L_{k,2}} \right) + \cdots \\
+ \left( \frac{L_{k,n+1}}{L_{k,n}} - \frac{L_{k,n}}{L_{k,n-1}} \right)
\]

\[
= \frac{L_{k,2}}{L_{k,1}} - \frac{(L_{k,2}^2 - L_{k,3}L_{k,1})}{L_{k,1}L_{k,2}} - \frac{(L_{k,3}^2 - L_{k,2}L_{k,4})}{L_{k,2}L_{k,3}} - \cdots \\
- \frac{(L_{k,n}^2 - L_{k,n+1}L_{k,n-1})}{L_{k,n-1}L_{k,n}}
\]

Using the equations (11) and (14)

\[
F_{k,n}^2 - F_{k,n-1}F_{k,n+1} = (-1)^{n-1}
\]

\[
L_{k,n}^2 - L_{k,n-1}L_{k,n+1} = (-1)^n \Delta
\]

Gives

\[
\frac{F_{k,n+1}}{F_{k,n}} = \frac{F_{k,2}}{F_{k,1}} - \frac{(-1)^2}{F_{k,1}F_{k,2}} - \frac{(-1)^3}{F_{k,2}F_{k,3}} - \cdots - \frac{(-1)^{n-1}}{F_{k,n-1}F_{k,n}}
\]
Taking limit as $\lim_{n \to \infty}$, gives

$$r_1 = \frac{1 + \sqrt{k^2 + 4}}{2} = k + \frac{1}{1. k} - \frac{1}{k (k^2 + 4)} + \ldots$$

For Fibonacci series

$$\frac{1 + \sqrt{5}}{2} = 1 + \frac{1}{1.1} - \frac{1}{1.2} + \frac{1}{2.3} - \frac{1}{3.5} + \frac{1}{5.8} - \frac{1}{8.13} + \ldots$$

Now, we obtain more general relation for $F_{k,n}$ and $L_{k,n}$ as a series of fractions:

**Theorem 4.2.** For $n, k > 0$,

1. 

$$\frac{F_{k,(n+1)t}}{F_{k,nt}} = \frac{F_{k,2t}}{F_{k,t}} - \frac{(-1)^t F_{k,2t}^2}{F_{k,2t} F_{k,3t}} - \frac{(-1)^{2t} F_{k,3t}^2}{F_{k,3t} F_{k,4t}} - \frac{(-1)^{3t} F_{k,4t}^2}{F_{k,4t} F_{k,5t}} - \ldots \frac{(-1)^{(n-1)t} F_{k,t}^2}{F_{k,(n-1)t} F_{k,nt}}$$

2. 

$$\frac{L_{k,(n+1)t}}{L_{k,nt}} = \frac{L_{k,t}}{L_{k,0}} + \frac{\Delta F_{k,t}^2}{L_{k,0} L_{k,t}} + \frac{(-1)^t F_{k,2t}^2}{L_{k,2t} L_{k,3t}} + \frac{(-1)^{2t} F_{k,3t}^2}{L_{k,3t} L_{k,4t}} + \ldots \frac{(-1)^{(n-1)t} F_{k,t}^2}{L_{k,(n-1)t} L_{k,nt}}$$
Proof. We can write expressions of \( \frac{F_{k(n+1)t}}{F_{kt}} \) and \( \frac{L_{k(n+1)t}}{L_{kt}} \) in series as

\[
\frac{F_{k(n+1)t}}{F_{kt}} = \frac{F_{k,2t}}{F_{kt}} + \left( \frac{F_{k,3t}}{F_{k,2t}} - \frac{F_{k,2t}}{F_{kt}} \right) + \left( \frac{F_{k,4t}}{F_{k,3t}} - \frac{F_{k,3t}}{F_{k,2t}} \right) + \ldots
\]

\[
= \frac{F_{k,2t}}{F_{kt}} - \frac{F_{k,2t}^2 - F_{k,3t}F_{k,t}}{F_{k,2t}F_{k,t}} - \frac{F_{k,3t}^2 - F_{k,2t}F_{k,4t}}{F_{k,3t}F_{k,2t}} - \ldots
\]

And

\[
\frac{L_{k(n+1)t}}{L_{kt}} = \frac{L_{k,t}}{L_{k,0}} + \left( \frac{L_{k,2t}}{L_{k,t}} - \frac{L_{k,t}}{L_{k,0}} \right) + \left( \frac{L_{k,3t}}{L_{k,2t}} - \frac{L_{k,2t}}{L_{k,t}} \right) + \ldots
\]

\[
= \frac{L_{k,t}}{L_{k,0}} - \frac{L_{k,0}L_{k,2t} - L_{k,t}L_{k,3t}}{L_{k,0}L_{k,t}} - \ldots
\]

Using the equations (15) and (16)

\[
F_{k,nt}^2 - F_{k,(n-1)t}F_{k,(n+1)t} = (-1)^{(n-1)t}(F_{k,t})^2 \tag{22}
\]

\[
L_{k,nt}^2 - L_{k,(n-1)t}L_{k,(n+1)t} = (-1)^{(n-1)t} \Delta(F_{k,t})^2 \tag{23}
\]
Godase, AD, Dhakne, MB

Gives

\[
\frac{F_{k,(n+1)t}}{F_{k,nt}} = \frac{F_{k,2t}}{F_{k,t}} - \frac{(-1)^1F_{k,t}^2}{F_{k,2t}F_{k,3t}} - \frac{(-1)^2F_{k,t}^2}{F_{k,3t}F_{k,4t}} - \frac{(-1)^3F_{k,t}^2}{F_{k,4t}F_{k,(n-1)t}} - \frac{(-1)^{(n-1)t}F_{k,t}^2}{F_{k,(n-1)t}F_{k,nt}}
\]

And

\[
\frac{L_{k,(n+1)t}}{L_{k,nt}} = \frac{L_{k,t}}{L_{k,0}} + \frac{\Delta F_{k,t}^2}{L_{k,0}L_{k,t}} + \frac{(-1)^1F_{k,t}^2}{L_{k,t}L_{k,2t}} + \frac{(-1)^2F_{k,t}^2}{L_{k,2t}L_{k,3t}} + \ldots - \frac{(-1)^{(n-1)t}F_{k,t}^2}{L_{k,(n-1)t}L_{k,nt}}
\]

\[\square\]

**Theorem 4.3.** For \(n, m, k > 0\),

1. 

\[
\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^nF_{k,t} \left[ \frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^1}{L_{k,n+t}L_{k,n+2t}} + \frac{(-1)^2}{L_{k,n+2t}L_{k,n+3t}} + \ldots + \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}} \right]
\]

2. 

\[
\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^nF_{k,t} \left[ \frac{1}{L_{k,n}L_{k,n+t}} + \frac{(-1)^1}{L_{k,n+t}L_{k,n+2t}} + \frac{(-1)^2}{L_{k,n+2t}L_{k,n+3t}} + \ldots + \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}L_{k,n+mt}} \right]
\]
Proof. We can write expressions of \( \frac{F_{k,n+mt}}{L_{k,n+mt}} \) and \( \frac{L_{k,n+mt}}{F_{k,n+mt}} \) in series as

\[
\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + \left( \frac{F_{k,n+t}}{L_{k,n+t}} - \frac{F_{k,n}}{L_{k,n}} \right) + \left( \frac{F_{k,n+2t}}{L_{k,n+2t}} - \frac{F_{k,n+t}}{L_{k,n+2t}} \right) + \ldots.
\]

\[
+ \left( \frac{F_{k,n+(m-1)t}}{L_{k,n+(m-1)t}} - \frac{F_{k,n}}{L_{k,n}} \right)
\]

\[
= \frac{F_{k,n}}{L_{k,n}} + \frac{(F_{k,n+1}L_{k,n} - F_{k,n}L_{k,n+1})}{L_{k,n}L_{k,n+1}} + \frac{(F_{k,n+2}L_{k,n+1} - F_{k,n+1}L_{k,n+2})}{L_{k,n+1}L_{k,n+2}}
\]

\[
+ \ldots.
\]

And

\[
\frac{L_{k,n+mt}}{F_{k,n+mt}} = \frac{L_{k,n}}{F_{k,n}} + \left( \frac{L_{k,n+t}}{F_{k,n+t}} - \frac{L_{k,n}}{F_{k,n}} \right) + \left( \frac{L_{k,n+2t}}{F_{k,n+2t}} - \frac{L_{k,n+t}}{F_{k,n+2t}} \right)
\]

\[
+ \ldots.
\]

\[
= \frac{L_{k,n}}{F_{k,n}} - \frac{(F_{k,n+1}L_{k,n} - F_{k,n}L_{k,n+1})}{L_{k,n}L_{k,n+1}} - \frac{(F_{k,n+2}L_{k,n+1} - F_{k,n+1}L_{k,n+2})}{L_{k,n+1}L_{k,n+2}}
\]

\[
+ \ldots.
\]

Using the equations (15) and (16)

\[
F_{k,n}^2 - F_{k,(n-1)}F_{k,(n+1)} = (-1)^{(n-1)}(F_{k,k})^2
\]

\[
L_{k,n}^2 - L_{k,(n-1)}L_{k,(n+1)} = -(1)^{(n-1)}(\Delta(F_{k,k}))^2
\]

Gives

\[
\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n \left( \frac{1}{L_{k,n}L_{k,n+1}} + \frac{(-1)^n}{L_{k,n+2}L_{k,n+3}} \right) + \ldots.
\]

...
Godase, AD, Dhakne, MB

\[
\frac{F_{k,n+mt}}{L_{k,n+mt}} = \frac{F_{k,n}}{L_{k,n}} + 2(-1)^n F_{k,t} \left[ \frac{1}{L_{k,n+1}L_{k,n+2t}} + \frac{(-1)^t}{L_{k,n+2t}L_{k,n+3t}} + \ldots + \frac{(-1)^{(m-1)t}}{L_{k,n+(m-1)t}} \right]
\]

5 CONCLUSIONS

Some new relationship between \( k \) Fibonacci and \( k \) Lucas sequences using continued fractions and series of fractions are derived, this approach is different and never tried in \( k \) Fibonacci sequence literature.

REFERENCES

Fibonacci and Lucas Sequences as Series of Fractions
